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THE EXPRESSION OF CONSTANT AND OF ALTERNATING CONTINUED FRACTIONS IN HYPERBOLIC FUNCTIONS

BY A. E. KENNELLY

IN the theory of electric telegraphic or of electric telephonic lines, it becomes necessary to consider successive fractions of the forms

$$\frac{1}{a}, \quad \frac{1}{b+\frac{1}{a}}, \quad \frac{1}{a+\frac{1}{b+\frac{1}{a}}}, \quad \frac{1}{b+\frac{1}{a+\frac{1}{b+\frac{1}{a}}}} \quad \text{etc.}$$

It is desirable to obtain a simpler expression of these forms, when they are extended to a large number of terms.

Alternating continued fractions. Let us then consider two infinite continued fractions:

$$F_\infty(a, b) = \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cdots}}}} \qquad F_\infty(b, a) = \cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cdots}}}}$$

These may be called *alternating continued fractions*, because the two quantities a and b recur alternately in the successive denominators. Denoting the successive convergents of these fractions by

$$F_1(a, b), \quad F_2(a, b), \quad F_3(a, b), \quad F_4(a, b), \text{ etc.,} \qquad \text{and}$$
$$F_1(b, a), \quad F_2(b, a), \quad F_3(b, a), \quad F_4(b, a), \text{ etc.,}$$

the successive values of the physical problem are

$$F_1(a, b), \quad F_2(b, a), \quad F_3(a, b), \quad F_4(b, a), \text{ etc.}$$

Assuming first that a and b are real numbers of the same sign, we introduce two auxiliary variables θ_1 and θ_2 defined by the equations:

$$\sinh\theta_1 = \frac{a+b}{4} + \frac{a-b}{4}\sqrt{1+\frac{4}{ab}},\quad \cosh\theta_1 = \frac{a-b}{4} + \frac{a+b}{4}\sqrt{1+\frac{4}{ab}},\qquad (1)$$

$$\sinh\theta_2 = \frac{a+b}{4} - \frac{a-b}{4}\sqrt{1+\frac{4}{ab}},\quad \cosh\theta_2 = \frac{b-a}{4} + \frac{a+b}{4}\sqrt{1+\frac{4}{ab}},\qquad (2)$$

so that $\sinh\theta_1 + \sinh\theta_2 = \dfrac{a+b}{2}$; $\cosh\theta_1 - \cosh\theta_2 = \dfrac{a-b}{2}$.

We also define μ and δ by the equations:

$$\mu = \frac{\theta_1+\theta_2}{2} = \sinh^{-1}\frac{\sqrt{ab}}{2} = \cosh^{-1}\sqrt{1+\frac{ab}{4}},\qquad (3)$$

$$\delta = \frac{\theta_1-\theta_2}{2} = \sinh^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right) = \cosh^{-1}\left(\frac{a+b}{2\sqrt{ab}}\right).\qquad (4)$$

In these equations the radicals are understood to have a positive sign, and consequently μ is always positive.

We now have:

$$F_1(a,b) = \frac{1}{a} = \frac{1}{\sqrt{ab}}\cdot\sqrt{\frac{b}{a}} = \frac{1}{2\sinh\mu}\sqrt{\frac{b}{a}}$$

$$= \frac{\cosh\mu}{\sinh 2\mu}\sqrt{\frac{b}{a}} = \frac{\cosh\mu}{\sinh 2\mu}(\cosh\delta - \sinh\delta),$$

$$F_2(a,b) = \frac{\sinh 2\mu}{\cosh 3\mu}\sqrt{\frac{b}{a}} = \frac{\sinh 2\mu}{\cosh 3\mu}(\cosh\delta - \sinh\delta),$$

$$F_3(a,b) = \frac{\cosh 3\mu}{\sinh 4\mu}\sqrt{\frac{b}{a}} = \frac{\cosh 3\mu}{\sinh 4\mu}(\cosh\delta - \sinh\delta).$$

In general,

$$F_n(a,b) = \frac{\sinh n\mu}{\cosh(n+1)\mu}(\cosh\delta - \sinh\delta) = \frac{\sinh n\mu}{\cosh(n+1)\mu}\sqrt{\frac{b}{a}}\ \text{if } n \text{ even};\qquad (5)$$

$$F_n(a,b) = \frac{\cosh n\mu}{\sinh(n+1)\mu}(\cosh\delta - \sinh\delta) = \frac{\cosh n\mu}{\sinh(n+1)\mu}\sqrt{\frac{b}{a}}\ \text{if } n \text{ odd}.\qquad (6)$$

These formulas may be proved by induction. Assuming the truth of the formula for $F_{n-1}(a, b)$, and the corresponding formula* for $F_{n-1}(b, a)$ in which a and b exchange places, we have, if n is odd,

$$F_n(a, b) = \frac{1}{a + F_{n-1}(b, a)} = \frac{1}{a + \sqrt{\frac{a}{b}} \cdot \frac{\sinh (n-1)\mu}{\cosh n\mu}}$$

$$= \frac{1}{\left\{\sqrt{ab} + \frac{\cosh \mu \sinh n\mu - \sinh \mu \cosh n\mu}{\cosh n\mu}\right\} \sqrt{\frac{a}{b}}}$$

$$= \frac{1}{(2 \sinh \mu + \cosh \mu \tanh n\mu - \sinh \mu) \sqrt{\frac{a}{b}}} = \frac{1}{(\sinh \mu + \cosh \mu \tanh n\mu) \sqrt{\frac{a}{b}}}$$

$$= \frac{1}{\frac{\sinh n\mu \cosh \mu + \cosh n\mu \sinh \mu}{\cosh n\mu} \sqrt{\frac{a}{b}}} = \frac{\cosh n\mu}{\sinh \{(n+1)\mu\}} \sqrt{\frac{b}{a}};$$

and if n is even,

$$F_n(a, b) = \frac{1}{a + F_{n-1}(b, a)} = \frac{1}{a + \sqrt{\frac{a}{b}} \cdot \frac{\cosh (n-1)\mu}{\sinh n\mu}}$$

$$= \frac{1}{\left\{\sqrt{ab} + \frac{\cosh \mu \cosh n\mu - \sinh \mu \sinh n\mu}{\sinh n\mu}\right\} \sqrt{\frac{a}{b}}}$$

$$= \frac{1}{(2 \sinh \mu + \cosh \mu \coth n\mu - \sinh \mu) \sqrt{\frac{a}{b}}} = \frac{1}{(\sinh \mu + \cosh \mu \coth n\mu) \sqrt{\frac{a}{b}}}$$

$$= \frac{1}{\frac{\cosh n\mu \cosh \mu + \sinh n\mu \sinh \mu}{\sinh n\mu} \sqrt{\frac{a}{b}}} = \frac{\sinh n\mu}{\cosh \{(n+1)\mu\}} \sqrt{\frac{b}{a}}.$$

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* It follows at once that $F_n(b, a) = \frac{\sinh n\mu}{\cosh(n+1)\mu} \sqrt{\frac{a}{b}} = \frac{\sinh n\mu}{\cosh(n+1)\mu} (\cosh \delta + \sinh \delta)$

or $F_n(b, a) = \frac{\cosh n\mu}{\sinh(n+1)\mu} \sqrt{\frac{a}{b}} = \frac{\cosh n\mu}{\sinh(n+1)\mu} (\cosh \delta + \sinh \delta)$

according as n is even or odd.

In none of these equations is either sinh $n\mu$ or cosh $n\mu$ equal to zero, since μ is a positive real number.

If we define the value of the infinite continued fraction as

$$F_\infty(a, b) = \lim_{n=\infty} F_n(a, b),$$

we have, since μ is positive,

$$F_\infty(a, b) = \epsilon^{-\theta_1} = \epsilon^{-(\mu+\delta)} = \sqrt{\frac{b}{a}} \cdot \epsilon^{-\mu} = \cosh\theta_1 - \sinh\theta_1$$

$$= \frac{b}{2}\left(\sqrt{1 + \frac{4}{ab}} - 1\right). \tag{7}$$

The last result may also be obtained by elementary **algebra**, which offers, however, no proof of the existence of the limit.

It may also be noted that $\dfrac{F_n(b, a)}{F_n(a, b)} = \dfrac{a}{b}$ whether n be odd or even. Similarly,

$$F_\infty(b, a) = \epsilon^{-\theta_2} = \epsilon^{-(\mu-\delta)} = \sqrt{\frac{a}{b}} \cdot \epsilon^{-\mu} = \cosh\theta_2 - \sinh\theta_2$$

$$= \frac{a}{2}\left(\sqrt{1 + \frac{4}{ab}} - 1\right). \tag{8}$$

Constant continued Fractions. If $b = a$, each of the alternating continued fractions $F_\infty(a, b)$ and $F_\infty(b, a)$ becomes identical with

$$F_\infty(a) = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cdots}}}$$

which we may call a *constant* continued fraction. In this case

$$\theta_1 = \theta_2 = \mu = \theta = \sinh^{-1}\left(\frac{a}{2}\right), \text{ and } \delta = 0.$$

The successive convergents then become:

$$F_1(a) = \frac{\cosh\theta}{\sinh 2\theta},$$

$$F_2(a) = \frac{\sinh 2\theta}{\cosh 3\theta},$$

$$F_3(a) = \frac{\cosh 3\theta}{\sinh 4\theta},$$

. . . . . .

$$F_n(a) = \frac{\sinh n\theta}{\cosh(n+1)\theta} \qquad \text{if } n \text{ is even,} \qquad (9)$$

$$= \frac{\cosh n\theta}{\sinh(n+1)\theta} \qquad \text{if } n \text{ is odd;} \qquad (10)$$

. . . . . .

$$F_\infty(a) = \epsilon^{-\theta}$$

$$= \cosh\theta - \sinh\theta = \frac{a}{2}\left(\sqrt{1+\frac{4}{a^2}} - 1\right). \qquad (11)$$

From the above expression for $F_\infty(a)$, it follows that any exponential value ϵ^{-m} may be regarded as the value of a constant continued fraction $F_\infty(2\sinh m)$.

Alternative Expression for a Constant Continued Fraction. The following alternative form for $F_n(a)$ is sometimes convenient.

$$F_1(a) = \frac{1}{\sinh\theta + \cosh\theta\tanh\theta} = \frac{1}{\sinh\theta+\sinh\theta} = \frac{1}{2\sinh\theta} = \frac{1}{a}, \qquad (12)$$

$$F_2(a) = \frac{1}{\sinh\theta + \cosh\theta\coth 2\theta},$$

$$F_3(a) = \frac{1}{\sinh\theta + \cosh\theta\tanh 3\theta},$$

. . . . . .

$$F_n(a) = \frac{1}{\sinh\theta + \cosh\theta\coth n\theta} \qquad \text{if } n \text{ is even,}$$

$$= \frac{1}{\sinh\theta + \cosh\theta\tanh n\theta} \qquad \text{if } n \text{ is odd;}$$

. . . . . .

$$F_\infty(a) = \frac{1}{\epsilon^\theta} = \epsilon^{-\theta} = \epsilon^{-\sinh^{-1}\left(\frac{a}{2}\right)} = \frac{1}{\sinh\theta+\cosh\theta} = \cosh\theta - \sinh\theta \qquad (13)$$

Consequently, as n increases from 1 to ∞, the second term in the denominator of (12) changes from $\sinh\theta$ to $\cosh\theta$ in (13).

Expression of any alternating continued fraction in terms of an equivalent constant continued fraction. It is evident on comparing equations (5) and (6) with (9) and (10) that the nth convergent of either of the alternating continued fractions $F(a, b)$ and $F(b, a)$ may be expressed in terms of the nth convergent of the constant continued fraction $F(\sqrt{ab})$, the constant term of which is the geometric mean of the two alternating terms in the alternating continued fraction. Thus:

$$F_5(a, b) = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a}}}}} = \sqrt{\frac{b}{a}} \cdot F_5(\sqrt{ab}) = \frac{b}{\sqrt{ab}} \times \cfrac{1}{\sqrt{ab} + \cfrac{1}{\sqrt{ab} + \cfrac{1}{\sqrt{ab} + \cfrac{1}{\sqrt{ab} + \cfrac{1}{\sqrt{ab}}}}}}$$

and

$$F_5(b, a) = \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b}}}}} = \sqrt{\frac{a}{b}} \cdot F_5(\sqrt{ab}) = \frac{a}{\sqrt{ab}} \times \cfrac{1}{\sqrt{ab} + \cfrac{1}{\sqrt{ab} + \cfrac{1}{\sqrt{ab} + \cfrac{1}{\sqrt{ab} + \cfrac{1}{\sqrt{ab}}}}}}$$

Particular cases of Alternating Continued Fractions. In the case of $b = \dfrac{a}{a + 1}$ in $F_\infty(b, a)$, we find by (2) that $\theta_2 = 0$ and $F_\infty(b, a) = 1$. Similarly, when $a = \dfrac{b}{b + 1}$, we find $\theta_1 = 0$ and $F_\infty(a, b) = 1$.

Case of a and b with opposite signs. If a and b have opposite signs, let $\sqrt{ab} = ic$, where c is a positive real number and $i = \sqrt{-1}$. Defining μ as in (3), we have

$$\sinh\mu = \frac{ic}{2},$$

whence,

$$\epsilon^{\mu} - \epsilon^{-\mu} = ic.$$

It follows, if we write $\mu = \nu + i\lambda$, that either $\nu = 0$, or $\lambda = \frac{\pi}{2}$ and ν is positive. If $\nu = 0$ we may show that

$$ab = -\,4\sin^2\lambda,$$

and hence that ab lies between 0 and $-\,4$. In this case the successive values of the convergents approach no limit, as may be readily verified. For all other cases

$$\mu = \nu + i\frac{\pi}{2}.$$

Substituting in (3) we obtain

$$\nu = \cosh^{-1}\left(\frac{c}{2}\right) = \sinh^{-1}\sqrt{\frac{c^2}{4} - 1}\,, \tag{14}$$

where again the radical has the positive sign, since ν is positive. Similarly, from (4),

$$\delta = \gamma \pm i\frac{\pi}{2}, \tag{15}$$

where γ is a real quantity, the sign of $i\frac{\pi}{2}$ following the sign of $(a - b)$ in (4). Substituting these values of μ and δ in equations (5) and (6) we find, whether n be odd or even:

$$F_n(a,\,b) = -\,i\,\frac{\sinh n\nu}{\sinh(n+1)\nu}\sqrt{\frac{b}{a}}\,, \tag{16}$$

and

$$F_\infty(a,\,b) = -\,i\sqrt{\frac{b}{a}}\;.\;\epsilon^{-\nu} = \frac{b}{2}\left(\sqrt{1 + \frac{4}{ab}} - 1\right). \tag{17}$$

It also follows that, whether n be odd or even,

$$F_n(b, a) = -i \frac{\sinh n\nu}{\sinh (n+1)\nu} \sqrt{\frac{a}{b}}, \qquad (18)$$

and

$$F_\infty(b, a) = -i\sqrt{\frac{a}{b}} \cdot \epsilon^{-\nu} = \frac{a}{2}\left(\sqrt{1 + \frac{4}{ab}} - 1\right). \qquad (19)$$

Case of $b = -a$, *and* $-ab = c^2 > 4$. If $b = -a$, $F(a, b)$ becomes

$$F(a, b) = \cfrac{1}{a + \cfrac{1}{-a + \cfrac{1}{a + \cfrac{1}{-a + \cdots}}}} = \cfrac{1}{a - \cfrac{1}{a - \cfrac{1}{a - \cfrac{1}{a - \cdots}}}}$$

so that a constant continued fraction with all its signs negative is equivalent to an alternating continued fraction with its two terms equal in magnitude but opposite in sign. In this case, whether n be odd or even,

$$F_n(a, -a) = \frac{\sinh n\nu}{\sinh (n+1)\nu}, \qquad (20)$$

and

$$F_\infty(a, -a) = \epsilon^{-\nu} = -\frac{a}{2}\left(\sqrt{1 - \frac{4}{a^2}} - 1\right). \qquad (21)$$

Since $F_n(-a, a) = -F_n(a, -a)$, the successive convergents of $F_n(-a, a)$ are found by taking the results in (20) and (21) with the opposite sign. Moreover, when $b = -a$, $F(\sqrt{ab})$ becomes $F(ic)$ or

$$F(ic) = \cfrac{1}{ic + \cfrac{1}{ic + \cfrac{1}{ic + \cfrac{1}{ic + \cdots}}}} = -i \times \cfrac{1}{c - \cfrac{1}{c - \cfrac{1}{c - \cfrac{1}{c - \cdots}}}} = -i \times \cfrac{1}{c + \cfrac{1}{-c + \cfrac{1}{c + \cfrac{1}{-c + \cdots}}}}$$

so that substituting in (20) and (21), whether n be odd or even,

$$F_n(ic) = -i \frac{\sinh n\nu}{\sinh (n+1)\nu} \qquad (22)$$

and

$$F_\infty(ic) = -i\epsilon^{-\nu} = \frac{ic}{2}\left(\sqrt{1-\frac{4}{c^2}}-1\right). \tag{23}$$

Case of a and b having opposite signs, with $0 < c^2 < 4$. In this case we may take $\nu = 0$, and write $\mu = i\lambda$, where λ is a positive real number. Then equations (16) and (17) become:

$$F_n(a, b) = -i\sqrt{\frac{b}{a}} \times \frac{\sin n\lambda}{\sin (n+1)\lambda}, \tag{24}$$

$$F_n(b, a) = -i\sqrt{\frac{a}{b}} \times \frac{\sin n\lambda}{\sin (n+1)\lambda}, \tag{25}$$

where $\lambda = \cos^{-1}\left(\frac{c}{2}\right) = \sin^{-1}\sqrt{1-\frac{c^2}{4}}$, whether n be odd or even. As already pointed out, these expressions do not converge to a limit.

Case of $b = -a$ *with* $0 < c^2 < 4$.

When $b = -a$ in (24) we have

$$F_n(a, -a) = \frac{\sin n\lambda}{\sin (n+1)\lambda} = \cfrac{1}{a - \cfrac{1}{a - \cfrac{1}{a - \cdots}}} \text{ to } n \text{ terms.} \tag{26}$$

and

$$F_n(-a, a) = \frac{-\sin n\lambda}{\sin (n+1)\lambda}, \tag{27}$$

whether n be odd or even. These expressions do not converge as n becomes infinite.

Equation (26) covering the particular case of a constant continued fraction with all its numerators negative after the first, and $0 < c^2 < 4$, expressed in terms of circular functions, was published by Strehlke in 1864.* Strehlke's formula does not, however, extend to $F_n(a)$, a constant continued fraction with positive signs, and still less does it apply to alternating continued fractions.

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* A. B. Strehlke, Grunert's *Archiv der Mathematik und Physik*, 1864, vol. 42, p. 343.

Terminally Loaded Alternating Continued Fractions. Among the physical applications already referred to, the following *ascending* series of alternating continued fractions present themselves:

$$F_1(a, b)_m = \cfrac{1}{a + m}$$

$$F_2(b, a)_m = \cfrac{1}{b + \cfrac{1}{a + m}}$$

$$F_3(a, b)_m = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + m}}} \quad \text{etc.}$$

Consistently with this notation, we may write

$$F_0(b, a)_m = m.$$

The constant quantity m appearing at the end of each expression in this series may be called the *terminal load* of the alternate continued fractions $F_n(a, b)_m$ and $F_n(b, a)_m$.

It may be demonstrated that:

$$F_n(a, b)_m = \sqrt{\frac{b}{a}} \cdot \frac{\sinh(n\mu + \phi)}{\cosh\{(n + 1)\mu + \phi\}} \qquad \text{if } n \text{ is even} \qquad (28)$$

$$= \sqrt{\frac{b}{a}} \cdot \frac{\cosh(n\mu + \phi)}{\sinh\{(n + 1)\mu + \phi\}} \qquad \ldots \text{ odd} \qquad (29)$$

$$F_n(b, a)_m = \sqrt{\frac{a}{b}} \cdot \frac{\sinh(n\mu + \phi)}{\cosh\{(n + 1)\mu + \phi\}} \qquad \ldots \text{ even} \qquad (30)$$

$$= \sqrt{\frac{a}{b}} \cdot \frac{\cosh(n\mu + \phi)}{\sinh\{(n + 1)\mu + \phi\}} \qquad \ldots \text{ odd} \qquad (31)$$

where μ is the same quantity as is defined in (3) or appears in equations (5) and (6) for the corresponding unloaded alternating continued fractions, and

ϕ is defined by the relation :

$$F_0(b, a)_m = m = \sqrt{\frac{a}{b}} \cdot \frac{\sinh \phi}{\cosh (\mu + \phi)}$$

or

$$\phi = \tanh^{-1} \frac{\cosh \mu}{\sqrt{\frac{a}{b}} \cdot \frac{1}{m} - \sinh \mu} \cdot \qquad (32)$$

If m be a positive real quantity between 0 and $\sqrt{\frac{a}{b}} \cdot \epsilon^{-\mu} = F_\infty (b, a)$, ϕ is real and positive. If m exceeds $\sqrt{\frac{a}{b}} \cdot \epsilon^{-\mu}$, we may write

$$\phi = \phi' + \frac{i\pi}{2} \qquad (33)$$

or

$$\phi' = \tanh^{-1} \left\{ \frac{\sqrt{\frac{a}{b}} \cdot \frac{1}{m} - \sinh \mu}{\cosh \mu} \right\}, \qquad (34)$$

where ϕ' is positive and real.

Substituting in equations (28) to (31), we obtain when m exceeds $\sqrt{\frac{a}{b}} \cdot \epsilon^{-\mu}$,

$$F_n(a, b)_m = \sqrt{\frac{b}{a}} \cdot \frac{\cosh(n\mu + \phi')}{\sinh \{ (n + 1)\mu + \phi' \}} \qquad \text{if } n \text{ is even} \qquad (35)$$

$$= \sqrt{\frac{b}{a}} \cdot \frac{\sinh(n\mu + \phi')}{\cosh \{ (n + 1)\mu + \phi' \}} \qquad \text{. . . odd} \qquad (36)$$

$$F_n(b, a)_m = \sqrt{\frac{a}{b}} \cdot \frac{\cosh(n\mu + \phi')}{\sinh \{ (n + 1)\mu + \phi' \}} \qquad \text{. . . even} \qquad (37)$$

$$= \sqrt{\frac{a}{b}} \cdot \frac{\sinh(n\mu + \phi')}{\cosh \{ (n + 1)\mu + \phi' \}} \qquad \text{. . . odd} \qquad (38)$$

In the particular case when $m = \frac{2}{b}$, $\phi' = 0$ and

$$F_n(a, b)_{\frac{2}{b}} = \sqrt{\frac{b}{a}} \cdot \frac{\cosh n\mu}{\sinh(n+1)\mu} \qquad \text{if } n \text{ is even} \tag{39}$$

$$= \sqrt{\frac{b}{a}} \cdot \frac{\sinh n\mu}{\cosh(n+1)\mu} \qquad \cdot\ \cdot\ \cdot \text{ odd} \tag{40}$$

$$F_n(b, a)_{\frac{2}{b}} = \sqrt{\frac{a}{b}} \cdot \frac{\cosh n\mu}{\sinh(n+1)\mu} \qquad \cdot\ \cdot\ \cdot \text{ even} \tag{41}$$

$$= \sqrt{\frac{a}{b}} \cdot \frac{\sinh n\mu}{\cosh(n+1)\mu} \qquad \cdot\ \cdot\ \cdot \text{ odd} \tag{42}$$

The corresponding expressions for terminally loaded constant continued fractions may be found from the preceding equations by putting $b = a$.

It is evident from the above conditions that continued fractions which are not constant may be expressed in hyperbolic functions. Thus the five-stage continued fraction

$$\cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{b}}}}} = F_3(a)_{F_2(b)}$$

may be expressed as the third approximation to a constant continued fraction, terminally loaded by the second approximation to another constant continued fraction.

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